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WAVEGUIDING AND ANOMALOUS PROPERTIES OF A PERIODIC KNIFE-TYPE GRATING

S. V. Sukhinin

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It is shown that periodic knife-type gratings always have waveguiding and anomalous properties. The dispersion relations for waveguide modes are obtained and the passbands are determined. The asymptotical form of the dispersion relations is obtained with an infinite increase in the length of the cascade elements and a decrease in the wave number. The effect of the geometrical characteristics and the type of grating on its waveguiding and anomalous properties is investigated.

INTRODUCTION

Waveguide-type generalized eigenfunctions describe the standing and traveling waves localized near a periodic structure. Studying the waveguide property is complicated by the fact that the corresponding self-conjugate extensions of the Laplace operator have a continuous spectrum.

The author [1] showed the existence of the waveguiding property of a periodic knife-type grating with large-sized elements using the analytical Fredholm theorem. This property of knife-type gratings is attributed to the eigenoscillations near the plate in a channel [2]. The results of approximate studies of the waveguiding property of the gratings with rather large-sized elements (compared with the period), dispersion relations, and the form of the waveguide functions were given in [3, 4].

The present work shows the existence of the waveguiding property for arbitrary knife-type gratings by the "Dirichlet–Neumann bracket" method [5].

Definition. The grating possessing both translational symmetry and the symmetry of the type of a second-order dihedral group D_2 is called a *simple knife-type grating* (type I in Fig. 1; the points having the D_2-type symmetry are crossed). The grating composed of two simple gratings with mutually intersecting interprofile channels and parallel elements of gratings (type II) is called a *combined knife-type grating.* The combined knife-type grating is called a *double* cascade if the interprofile channels are not intersected (type III).

1. FORMULATION AND SYMMETRY OF THE PROBLEM

Let G be the profile of a knife-type grating of plates on the plane of the Cartesian coordinates $\{x, y\} = R^2$ and $\Omega = R^2 \backslash G$ be the region of oscillations. It is assumed that G and Ω are unidirectionally periodic along the y axis with period 1. If G is a simple knife-type grating, the coordinates of the profiles on the y axis are integers. Let $\Omega_1 = \{(x,y){:}\ -L/2 \leqslant x \leqslant L/2, 0 < y < 1\}$, $\Omega_2 = \{(x,y){:}\, L/2 \leqslant x, 0 \leqslant y \leqslant 1\}$, $\Omega_3 = \{(x,y){:}\, x \leqslant -L/2, 0 \leqslant y \leqslant 1\}$, and L be the length of the plate profile. The regions $\{(x,y){:}\ -L/2 \leqslant x \leqslant L/2,\ k < y < k+1\}$ are called the interprofile channels (k is an integer). Restricting the function $u(x,y)$

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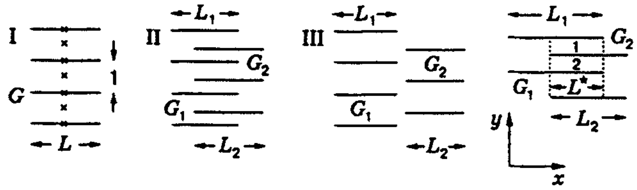

Fig. 1

on the domain Ω_j is denoted by $u_j(x,y)$ ($j = 1, 2, 3$). All variables are dimensionless: the spatial variables are related to the natural grating period H, and the temporal variable to the characteristic time H/c, where c is the velocity of the signal. The origin of the coordinates is chosen in the center of one of the grating elements, and the ordinate is parallel to the direction of the grating periodicity. The notation is shown in Fig. 1 (on the right).

The steady-state oscillations near the cascade are described by the function $u(x,y)$. In the region of oscillations Ω, it satisfies the wave equation

$$u_{xx} + u_{yy} + \lambda^2 u = 0. \tag{1.1}$$

Here λ means the dimensionless frequency of oscillations and it is assumed that $\lambda \geqslant 0$. If ω, L_g, and H are the real circular frequency of oscillations, the real length of the grating element, and the real grating period, respectively, then the expressions $\lambda = \omega H/c$ and $L = L_g/H$ are true for the dimensionless parameters and the grating period is equal to 1. The no-flow condition should be satisfied on the grating elements G:

$$\frac{\partial u}{\partial n} = 0, \tag{1.2}$$

where n is the normal to the surface of the grating element. In any bounded domain Ω_b, which is a subdomain of Ω, the local-finiteness condition for the energy of oscillations should be valid:

$$E(u, \Omega_b) = \int_{\Omega_b} [u^2 + (\nabla u)^2]\, d\Omega_b < \infty. \tag{1.3}$$

We call relations (1.1)–(1.3) Problem B. Because the Laplace operator is invariant under all motions of the plane R^2, the symmetry of the grating determines the symmetry of the corresponding boundary-value problem.

Definition 1.1. A structure is *unidirectionally periodic* if it admits a group of locally plane symmetries containing a subgroup T of translations parallel to a certain vector.

Since the group of symmetries of the unidirectionally periodic structure necessarily contains the subgroup T of translations along a certain y axis, only the following nontrivial subgroups of the group of admissible symmetries are possible: D_1 is the dihedral group with one axis of mirror symmetry (two types of mirror symmetries are possible: D_1^x is the axis of mirror symmetry parallel to the x axis and D_1^y is the axis of mirror symmetry parallel to the y axis), D_2 is the dihedral group with two axes of mirror symmetry, C_2 is the group of rotations on π, and T_σ is the group of sliding symmetries.

Knife-type gratings may be classified based on the groups of admissible symmetries. The simple knife-type grating has a maximum symmetry; it has points of symmetry of D_2 type located at a distance of 1/2 from each other (crosses in Fig. 1). The combined and double gratings can have points of C_2-type symmetry located at a distance of 1/2 from each other if all the profiles are the same in dimensions. They usually have only the translational symmetry.

The subgroups of symmetry of the grating allow one to divide the space of admissible solutions of Problem B into the corresponding invariant subspaces, which simplifies the analysis. The invariance of the problem relative to the mirror symmetry allows one to divide the space of admissible solutions into even and odd functions relative to the axis of symmetry. We used the invariance of the spaces of solutions of Problem B relative to the transformation D_1^y, in which the axis of mirror symmetry passes through the centers of all the elements of the simple knife grating. In this case, the space of solution of Problem B can be divided into a direct sum of two subspaces consisting of the symmetrical (even) $u^{(+)}(x,y) = D_1^y u^{(+)}(x,y) = u^{(+)}(-x,y)$ and antisymmetrical (odd) $u^{(-)}(x,y) = -D_1^y u^{(-)}(x,y) = -u^{(-)}(-x,y)$ functions in the variable x.

As the group of translations T is commutative and its representation $\hat{T}$ in the space of admissible solutions of Problem B is unitary, the space of solutions can be divided into one-dimensional subspaces invariant relative to the group $\hat{T}$. The functions $u(x,y)$, which belong to these subspaces, satisfy the conditions

$$u(x,y+1) = e^{i\xi}u(x,y) \tag{1.4}$$

and, as a result, have the form $u(x,y) = e^{i\xi y}v(x,y)$, $v(x,y+1) = v(x,y)$ in the free domain of oscillations. Hereafter, i is an imaginary unity and ξ is an arbitrary parameter $(-\pi < \xi \leqslant \pi)$ which describes the shift of the oscillation phase in the adjacent fundamental areas of the group of translations. Below, we call Problem B with condition (1.4) Problem $B(\xi)$. Sometimes the representation of the form (1.4) is called the Floquet theorem or Rayleigh–Bloch waves.

Remark 1.1. By virtue of the translational symmetry, it suffices to consider Problem $B(\xi)$ in the domain $\{(x,y) : 0 \leqslant y \leqslant 1\}$ bounded relative to y, i.e., in the band. Conditions (1.4) are satisfied in the entire domain of oscillations $\Omega = R^2\backslash G$ and allow one to extend the solutions to the entire plane from any fundamental domain of the group of translations.

With other conditions not being specified, Problem $B(\xi)$ is considered for a simple knife-type grating in the domain $\Omega_0 = \Omega_1 \cup \Omega_2 \cup \Omega_3$ for the values of the parameter ξ in the half-interval $0 < \xi \leqslant \pi$. The self-conjugate extensions of the Laplace operator in the domain $L_2(\Omega)$ with condition (1.2) or the corresponding restrictions are considered.

Definition 1.2. The generalized eigenfunctions of Problem $B(0)$, localized in the neighborhood of a knife-type grating, are called *anomalous functions* or solutions of this problem, and the corresponding frequencies are called *anomalous frequencies.*

By virtue of the symmetry of Problem $B(0)$ for a simple knife-type grating, if $u^*(x,y)$ is a generalized eigenfunction, the functions $u^*(-x,y)$, $u^*(-x,-y)$, $u^{\text{sym}}(x,y) = u^*(x,y) + u^*(x,-y)$, and $u^{\text{asym}}(x,y) = u^*(x,y) - u^*(x,-y)$ are its generalized eigenfunctions. The relations $u^{\text{sym}}(x,y) = u^{\text{sym}}(x,-y)$ (the parity on the ordinate) and $u^{\text{asym}}(x,y) = -u^{\text{asym}}(x,-y)$ (the oddness on the ordinate) are true. It is noteworthy that the mirror map relative to the abscissa $(x,y) \to (x,-y)$ is a mirror map relative to the plate located in the coordinate origin. The lemma below is true.

Lemma 1.1. *There are no nontrivial anomalous functions of Problem* $B(0)$ *which are even relative to the axispassing through the grating element.*

Proof. Let $u^{\text{sym}}(x,y)$ be the anomalous function of Problem $B(0)$ which is even relative to the abscissa, and $u^{\text{sym}}(x,y) = u^{\text{sym}}(x,-y)$. Since the function $u^{\text{sym}}(x,y)$ is periodic in the variable y with period 1, it is even relative to the axis containing any element of the grating. This means that the function $u^{\text{sym}}(x,y)$ is continuous in the entire domain and is a weak solution of Eq. (1.1) and a smooth solution by virtue of the theory of elliptical equations. Lemma 1.1 is proved.

Lemma 1.2. *The anomalous function of Problem* $B(0)$ *is odd relative to each axis parallel to the elements of the grating and passing through the point of symmetry of the dihedral group of this problem.*

Proof. Let $u^{\text{asym}}(x,y)$ be the anomalous function of Problem $B(0)$ which is odd relative to the abscissa and $u^{\text{asym}}(x,y) = -u^{\text{asym}}(x,-y)$. Because the function $u^{\text{asym}}(x,y)$ is periodic in y with period 1, the relations

$$u^{\text{asym}}(x,y+1) = u^{\text{asym}}(x,y) = -u^{\text{asym}}(x,-y-1) \tag{1.5}$$

are true.

The function $u^{\text{asym}}(x,-y-1)$ is the mirror reflection of the function $u^{\text{asym}}(x,y)$ relative to the $y = 1/2$ axis and, therefore, relations (1.5) means that it is odd relative to this axis. The statement of Lemma 1.2 holds true by virtue of the periodicity of the anomalous function with period equal to the minimum period of the grating in y.

The statements formulated in Lemmas 1.1 and 1.2 allow us to classify the modes of oscillations relative to the groups of admissible symmetries and to indicate the signatures of oscillations of the waveguide modes α and β and the anomalous modes γ and δ in the interprofile channel (Fig. 2). Any waveguide (or anomalous)

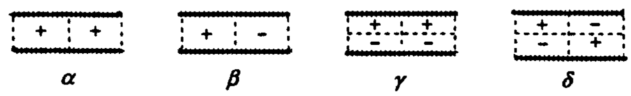

Fig. 2

function $u(x, y)$ can be presented as a sum of the modes α and β (γ and δ): $u(x, y) = u^{(\alpha)}(x, y) + u^{(\beta)}(x, y)$ [or $u(x, y) = u^{(\gamma)}(x, y) + u^{(\delta)}(x, y)$].

2. WAVEGUIDING PROPERTY

Definition 2.1. A generalized eigenfunction of the corresponding self-conjugate extension of the Laplace operator for Problem $B(\xi)$ is called a *waveguide* function of this problem if it is localized in the neighborhood of the grating of plates G. A generalized eigenfunction which is not waveguide is called a *free function*. The corresponding frequency of oscillations is called a waveguide or free frequency.

The unidirectionally periodic structure G has the waveguide property if a nontrivial waveguide function of Problem $B(\xi)$ exists. The general solution of Eq. (1.1) with the quasi-periodicity condition (1.4) in the free space has the form

$$u(x, y) = \sum_{n=-\infty}^{+\infty} \exp\left[i(2\pi n + \xi)y\right]\left[b_n^{+} \exp\left(|x|\beta_n\right) + b_n^{(-)} \exp\left(-|x|\beta_n\right)\right],$$

where $\beta_n = \sqrt{(2\pi n + \xi)^2 - \lambda^2}$. Therefore, the generalized eigenfunction $u(x, y)$ is localized in the neighborhood of the grating only in the case where the representation

$$u(x, y) = \sum_{n=-\infty}^{+\infty} b_n^{(\pm)} \exp\left[i(2\pi n + \xi)y\right] \exp\left(-|x|\beta_n\right) \tag{2.1}$$

is true in the free space with $\mathrm{Re}\,(\beta_n) > 0$ for all terms.

Remark 2.1. If there is a bounded generalized eigenfunction of Problem $B(\xi)$ of the form (2.1) for a certain λ_* ($0 < \lambda_* < \xi$), this function is a waveguide function. The set of waveguide frequencies of Problem $B(\xi)$ is discrete in the entire set of real numbers [1] and, hence, finite on the interval $0 < \lambda < \xi$.

Form of the Waveguide Function. For even and odd oscillations in the variable x (of α and β modes, respectively), the general representation of the waveguide solution of Problem $B(\xi)$ has the form

$$u_1(x, y) = a_0 \left\{ \begin{array}{c} \cos(\lambda x) \\ \sin(\lambda x) \end{array} \right\} + \sum_{m=1}^{+\infty} a_m \cos(m\pi y) \left\{ \begin{array}{c} \cosh(x\alpha_m) \\ \sinh(x\alpha_m) \end{array} \right\},$$

$$u_2(x, y) = \sum_{n=-\infty}^{+\infty} b_n \exp\left[i(2\pi n + \xi)y\right]\left[\exp\left(-|x|\beta_n\right)\right], \tag{2.2}$$

$$u_3(-x, y) = u_2(x, y) \quad \text{or} \quad u_3(-x, y) = -u_2(x, y),$$

where $\alpha_m = \sqrt{(\pi m)^2 - \lambda^2}$ and $\alpha_0 = i\lambda$.

For the functions $u(x, y)$ of the form (2.2) to be the solutions of Problem $B(\xi)$ at the boundaries of the domains Ω_j, where $j = 1, 2, 3$, the continuity conditions for the solution and its normal derivative should be satisfied. By virtue of the symmetry of the problem (for a simple knife-type grating), it suffices that these conditions be satisfied at one boundary, for example, at the boundary of the domains Ω_1 and Ω_2 ($x = L/2$ and $0 \leqslant y \leqslant 1$):

$$u_1 = u_2, \quad \frac{\partial u_1}{\partial x} = \frac{\partial u_2}{\partial x} \quad \text{for} \quad x = L/2. \tag{2.3}$$

Conditions (2.3) mean that the function of the form (2.2) is a weak solution of Problem $B(\xi)$, which is a strong solution of this problem by virtue of the theory of elliptical equations.

We used the "Dirichlet–Neumann bracket" method to study the existence of waveguide and anomalous frequencies [5]. Let the Dirichlet conditions (D) $u(x,y) = 0$ for $|x| = R > L/2$ or the Neumann conditions (N) $u_y(x,y) = 0$ for $|x| = R > L/2$ be satisfied on the $\{(x,y) : x = R\}$ and $\{(x,y) : x = -R\}$ lines, where $0 < R < \infty$, in addition to the boundary conditions of Problem $B(\xi)$.

Problem $B(\xi)$ with the additional condition D is denoted by $B(\mathrm{DR})$ and with the condition N by $B(\mathrm{NR})$. Let $\lambda_{\mathrm{DR}}^{(k)}$ be the eigenfrequency of Problem $B(\mathrm{DR})$, $\lambda_{\mathrm{NR}}^{(k)}$ be the eigenfrequency of Problem $B(\mathrm{NR})$, and $\lambda^{(k)}(\xi)$ be a certain eigenfrequency of Problem $B(\xi)$; the frequencies are numbered in ascending order: $k = 1, 2, \dots, K$. Since condition N expands the space of admissible solutions of Problem $B(\xi)$, and condition D narrows this domain, for all numbers $R > L/2$ the inequalities

$$\lambda_{\mathrm{NR}}^{(k)} < \lambda^{(k)}(\xi) < \lambda_{\mathrm{DR}}^{(k)} \qquad (k = 1, \dots, K) \tag{2.4}$$

are true, which can be obtained by means of the variational statement of the problem [3].

Remark 2.2. If the strict inequalities $0 < \lambda_{\mathrm{NR}}^{(k)}$ and $\lambda_{\mathrm{DR}}^{(k)} < \xi$ are fulfilled for $k = K$ at some values of $R \geqslant L/2$ and K, it follows from relation (2.4) that no less than K waveguide eigenvalues of Problem $B(\xi)$ exist.

Existence of the Waveguide Property. If $R = L/2$, condition D is a "soft" radiation condition. In this case, the first dimensionless eigenfrequency of longitudinal oscillations is $\lambda_{\mathrm{DL}/2}^{(1)}(\xi) = \pi/L$. Therefore, to satisfy the inequality $\lambda_{\mathrm{DL}/2}^{(1)}(\xi) < \xi$, it suffices that $\pi/L < \xi$. The inequality is satisfied if the dimensionless length of the profile satisfies the inequality $\pi/\xi < L$. As conditions (1.4) are true, for $R > L/2$ the rigorous inequality $\lambda_{\mathrm{NR}} > 0$ holds true; it follows that the eigenvalue of Problem $B(\xi)$ exists and is strictly greater than zero if $L > \pi/\xi$. A more general statement is also true.

Theorem 2.1 (existence of the waveguide property). *A simple knife-type grating always possesses the waveguide property.*

Proof. It suffices to show that, for any value of $L > 0$, there is $R > 0$ such that the following inequalities are true:

$$0 < \lambda_{\mathrm{NR}}^{(1)} < \lambda_{\mathrm{DR}}^{(1)} < \xi. \tag{2.5}$$

Estimate from Below. If $R > L/2$, the domain $\{(x,y){:} -R < y < R\}\backslash G$ is connected and the solution of Problem $B(\mathrm{NR})$ cannot be a constant by virtue of condition (1.4) and, hence, $\lambda_{\mathrm{NR}}^{(1)} > 0$.

Estimate from Above. Any solution $u(x,y)$ of the problems $B(\xi)$ in the domain Ω has the form

$$u = u_{\mathrm{discont}} + u_{\mathrm{cont}}, \tag{2.6}$$

where u_{discont} is a discontinuous function in the set G which describes a simple knife-type grating and u_{cont} is a function continuous in the domain $\Omega \cup G$. The function u_{discont} is continuous at the edges of the profiles G; therefore, one can check that $u_{\mathrm{discont}} \equiv 0$ in the domains Ω_2 and Ω_3. Using a certain finite and continuous function $f(x)$ ($f(x) \equiv 0$ for $L/2 \leqslant |x|$) such that the local-finiteness condition for the energy (1.3) is satisfied, one can represent u_{discont} (in the interprofile channel Ω_1) as

$$u_{\mathrm{discont}} = \left\{ \begin{array}{lll} f(x), & |x| < L/2, & 0 < y < 1 \\ 0, & |x| > L/2, & 0 < y < 1 \end{array} \right\}. \tag{2.7}$$

In other interprofile channels, this function is found from (2.7) by multiplying by $\exp(i\xi)$ in the corresponding degree. Let the component u_{cont} in the representation (2.6) have the form $u_{\mathrm{cont}}(x,y) = \exp(i\xi y)\cos(\pi x/2R)$. This function satisfies Problem $B(\mathrm{DR})$, except for the boundary condition (1.2), and is continuous in $\Omega \cup G$. The function u_{discont} of the form (2.7) is representable as $f(x) = æ\cos(\pi x/L)$, where æ is a constant. In other interprofile channels, the form of u_{discont} is determined by relation (1.4). For all values of æ, the relation which reflects the variational property of the eigenvalues is true [3]:

$$(\lambda_{\mathrm{DR}})^2 \leqslant \int\limits_{\Omega_R} [\nabla(u_{\mathrm{cont}} + u_{\mathrm{discont}})]^2 \, d\Omega_R \Big/ \int\limits_{\Omega_R} (u_{\mathrm{cont}} + u_{\mathrm{discont}})^2 \, d\Omega_R = \mu^2(æ, R). \tag{2.8}$$

Hereafter $\Omega_R = \Omega_0 \cap \{(x,y):|x| \leqslant R\}$. It is checked by a direct evaluation (this is the consequence of the bounded support of the function u_{discont} in the domain Ω_1) that, for large values of R, the asymptotical representation $\mu^2(æ, R) \cong \xi^2 + A/R + B/R^2$ is true. The quantities A and B are æ-dependent. Since the parameters R and æ are independent, the value of A is determining for fairly large R. The expression $A = (\pi^2 - \xi^2 L^2)æ^2/2L - 4L\xi \sin(\xi)æ/\pi$ holds true. Hence, the values of A are expected to be negative for small æ. Therefore, for large R and small positive æ, the strict inequality $\mu^2(æ, R) < \xi^2$ is true. By virtue of this relation, inequalities (2.5) hold true. Theorem 2.1 is proved.

Existence of the Waveguide Property for Combined and Double Knife-Type Gratings.

Definition 2.2. Let G_1 and G_2 be simple knife-type gratings with the same spatial phase and orientation (Fig. 1), $G = G_1 \cup G_2$, the combination of the gratings, and $[a_1, b_1]$ and $[a_2, b_2]$ be the projections of the elements G_1 and G_2 onto the x axis. The quantity $L^* = (b - a)$, where $[a, b] = [a_1, b_1] \cap [a_2, b_2]$, is the length of the shared part of the gratings. The parts of these gratings with coordinates $[a, b]$ on the abscissa are called a common part of the gratings G_1 and G_2.

The grating is combined if $L^* \neq 0$ and double otherwise. The combined and double gratings may be considered as the perturbation of the simple knife-type grating G_1 introduced by additional elements of the grating G_2. For Problem $B(\xi)$ with boundary conditions on G_1, this means that the space of admissible solutions is extended: the discontinuities of the solution are possible on the segments G_2. For combined and double gratings, a statement which is a consequence of Theorem 2.1 holds true.

Theorem 2.2 (the existence of the waveguide property in composite and double gratings). *The combined and double knife-type gratings always have the waveguiding property.*

Number of Waveguide Modes. Using relations (2.5), one can estimate the number of waveguide modes for each value of the parameter ξ, which describes the shift of the phase of oscillations in the adjacent interprofile channels, $0 < \xi \leqslant \pi$.

Theorem 2.3. *For each fixed value of ξ ($0 < \xi \leqslant \pi$) such that $\xi L/\pi$ is not an integer, the number $K(\xi)$ of waveguide modes describing the waves which run in one direction satisfies the relations* $\max(1, [\xi L/\pi]) \leqslant K(\xi) \leqslant [\xi L/\pi]+1$. *The greatest number $K_{\max} = K(\xi = \pi)$ of waveguide modes of Problem $B(\xi)$ which describe the waves running in one direction satisfies the relation $[L] \leqslant K_{\max} \leqslant [L]+1$. Here $[\ ]$ is the integer part of the corresponding number. If $\xi L/\pi$ is an integer, we have $(\xi L/\pi - 1) \leqslant K(\xi) \leqslant \xi L/\pi$ and $(L-1) \leqslant K_{\max} \leqslant L$.*

Proof. Let $R = L/2$ and $\lambda_{\text{NR}}^{(k)}$ and $\lambda_{\text{DR}}^{(k)}$ be the eigenvalues of problem $B(\xi)$ for the domain Ω_1 with the Neumann or Dirichlet conditions on the boundary $|x| = L/2$, respectively. The following representations are true:

$$\{\lambda_{\text{NR}}^{(k)}\}_{k=1,\ldots,(K+1)} = \{0, \pi/L, \ldots, (K-1)\pi/L, K\pi/L \geqslant \xi\}, \quad \{\lambda_{\text{DR}}^{(k)}\}_{k+1,\ldots,K} = \{\pi/L, \ldots, K\pi/L\}.$$

As the number of the waveguide modes on the interval of admissible frequencies $(0, \xi)$ cannot be greater than K^* such that $\lambda_{\text{NR}}^{(K^*)} < \xi \leqslant \lambda_{\text{NR}}^{(K^*+1)}$ and is not less than K_* such that $\lambda_{\text{DR}}^{(K_*)} < \xi \leqslant \lambda_{\text{DR}}^{(K_*+1)}$, the fact that was to be shown follows from this representation.

Dispersion Relations. Passbands. The parameter ξ from relation (1.4) may be regarded as a wave number for the waveguide function of a knife-type grating. The dimensionless waveguide frequencies λ depend on ξ; these dependences are the dispersion relations. The approximate dispersion relations for the elements of a large-sized grating were given in [4]; here we give them in a refined form.

If one takes into account the form of the waveguide function in the free space and the interprofile channel, conditions (2.3) are equivalent to the equalities

$$b_n = \exp(\beta_n L/2)\Big\{\int_0^1 u_1(L/2, y) \exp[-iy(\xi + 2\pi n)]\, dy\Big\},$$

$$b_n = -\frac{\exp(\beta_n L/2)}{\beta_n}\Big\{\int_0^1 \frac{\partial u_1(L/2, y)}{\partial x} \exp[-iy(\xi + 2\pi n)]\, dy\Big\}$$

for $\beta_n \neq 0$, $n = 0, \pm 1, \pm 2, \ldots$.

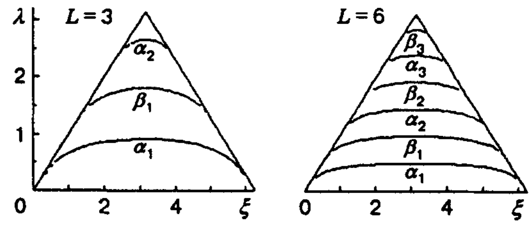

Fig. 3

As a result, we obtain an infinite homogeneous set of equations for desired quantities $\{a_m\}$ (m are integers), which define the function u_1:

$$\left\{\int_0^1 \left[u_1(L/2,y)+\frac{1}{\beta_n}\frac{\partial u_1(L/2,y)}{\partial x}\right]\exp\left[-iy(\xi+2\pi n)\right]dy\right\}=0, \qquad n=0,\pm1,\pm2,\ldots. \tag{2.9}$$

After the integration, the change of the desired variables, and the simplification, the homogeneous system (2.9) takes the following form relative to the quantities $\{\tilde{a}_m\}$:

$$\left\{\sum_{m=0}^{+\infty}\left[\frac{1}{\beta_n-\alpha_m}+\frac{\exp(-L\alpha_m)}{\beta_n+\alpha_m}\right]\tilde{a}_m=0\right\}.$$

This system has the canonical form, and one can study it by the method of expansion of the determinant [4]. The consequence is the dispersion relation which is true for all waveguide modes:

$$\sin\left[\Theta(\lambda,\xi)\right]=0. \tag{2.10}$$

Here

$$\Theta(\lambda,\xi)=\Bigg\{\lambda L-2\frac{\lambda\ln(2)}{\pi}-\frac{\lambda}{\pi}\Bigg[\Psi\Big(\frac{2N\pi+2\pi+\xi}{2\pi}\Big)+\Psi\Big(\frac{2N\pi+2\pi-\xi}{2\pi}\Big)$$
$$-2\Psi(2N+1)-\Psi\Big(\frac{2\pi+\xi}{2\pi}\Big)-\Psi\Big(\frac{2\pi-\xi}{2\pi}\Big)+2\Psi\Big(\frac{\xi}{2\pi}\Big)+\cot\Big(\frac{\xi}{2}\Big)\Bigg]+2\arcsin\Big(\frac{\lambda}{|\xi|}\Big)$$
$$+2\sum_{n=1}^{N}\left[\arcsin\Big(\frac{\lambda}{2n\pi+\xi}\Big)+\arcsin\Big(\frac{\lambda}{2n\pi-\xi}\Big)\right]-2\sum_{n=1}^{2N}\left[\arcsin\Big(\frac{\lambda}{n\pi}\Big)\right]-\frac{2\lambda}{\xi}\Bigg\};$$

N is a natural number and $\Psi(\ldots)$ is the logarithmic derivative of the gamma-function. The dispersion relation (2.10) was investigated numerically.

Waveguide Modes. Passbands. One can describe the wave motion along the direction of the periodicity of the grating (along the y axis in our case) by a function of the form $u(x,y,t)=u^*(x,y)\exp(-i\lambda t)$, where $u^*(x,y)$ is a waveguide function of Problem $B(\xi)$. According to (1.4), in the free space we have $u(x,y,t)=v^*(x,y)\exp\left[i(\xi y-\lambda t)\right]$. The function $v^*(x,y)$ is periodic in the variable y, i.e., $v^*(x,y+1)=v^*(x,y)$, and is localized in the neighborhood of the knife-type grating. It may be considered to be the complex amplitude of the corresponding waveguide mode. This viewpoint allows us to consider the waveguide modes as the wave running along the direction of the periodicity and to determine the lengths L_w and the phase velocities C_{phase} of the waves described by the waveguide modes: $L_w=2\pi/\xi$ and $C_{\text{phase}}=\lambda/\xi$.

The dispersion relation (2.10) implicitly determines the dimensionless waveguide frequency as a function of the wave number for each waveguide mode: $\lambda_k=\lambda_k(\xi)$, $k=1,\ldots,K_{\max}$. Figure 3 shows diagrams of these functions for various lengths of the elements of the knife-type grating ($L=3$ and 6) and the lines $\lambda=\xi$ and $\lambda=2\pi-\xi$. The range of variation of the wave number ξ is chosen from 0 to 2π; in the range $(-\pi,0)$, the dispersion relations are determined using the symmetries and periodicity of the problem with respect to ξ.

Statement 2.1. For any length L of the elements of the knife-type grating, there is a finite number of dispersion relations corresponding to the waveguide modes. The number $K_{\max}$ of dispersion relations is determined by Theorem 2.3 for waves running in the same direction.

Theorem 2.4. *For any length L of the elements of a knife-type grating, there are a finite number of passbands, which are determined by the half-interval* $\sigma_1 = (0, \lambda_1^{\max}]$ *and the segments* $\sigma_k = \{\lambda : (k-1)\pi/L < \lambda_k^{\min} \leqslant \lambda \leqslant \lambda_k^{\max} < k\pi/L\}$, *where* $k = 2, \ldots, K_{\max}$. *The passbands with odd numbers correspond to the* α *modes, and those with even numbers to the* β *modes* ($\sigma_{2k-1} \leftrightarrow \alpha_k$ *and* $\sigma_{2k} \leftrightarrow \beta_k$, *where* $k = 1, \ldots, [K_{\max}/2]$).

The upper and lower boundaries of the passbands as functions of the length of the grating element can be found using the dispersion relation (2.10). The upper boundaries are determined by the substitution $\xi = \pi$, and the lower boundaries by the substitution $\xi = \lambda$ in (2.10).

Figure 4 shows the width and number of passbands of a simple knife-type grating versus its geometrical parameters for the corresponding waveguide modes.

The above statements supplement, update, and agree with the results obtained by other methods and at the "physical" level of rigor [4].

The consequence of the symmetry of the problem and hence the dispersion relations relative to the wave number ξ with respect to the point $\xi = \pi$ is the fact that the group velocity C_{group} of the waves described by the waveguide modes is zero ($C_{\text{group}} = 0$) at the point $\xi = \pi$ for all the waveguide modes. Therefore, the unidirectional periodic knife-type grating has $K_{\max}$ resonance frequencies $\lambda_k = \lambda_k(\pi)$, $k = 1, \ldots, K_{\max}$. This is an important difference between the knife-type grating and the discrete periodic structure of mass–spring type.

Asymptotical Representation of the Dispersion Relations for the First Waveguide Modes. For an infinite lengthening of the grating elements, relations (2.10) allow one to determine the behavior of the waveguide frequencies $\lambda^{(k)} = \lambda^{(k)}(\xi, L)$ of Problems B for the first waveguide modes ($k = 1, 2, \ldots$) in the neighborhood of zero. The expressions

$$\lambda^{(k)}(\xi, L) = \frac{k\pi^2 \tan(\xi/2)}{L\pi \tan(\xi/2) - \pi - 2\Psi(\xi/2\pi)\tan(\xi/2) - 2\ln(2)\tan(\xi/2) - 2\gamma\tan(\xi/2)},$$

$$\lambda^{(k)}(\xi = \pi, L) = \frac{k\pi^2}{L\pi + 2\ln(2)}$$

are true (γ is the Euler constant and $L \gg 1$).

For the first waveguide mode, at $\xi \ll 1$ and $L > 1$ the consequence of (2.10) is the expression $\lambda^{(1)}(\xi) = \xi - (L\pi - 2\ln(2))^2\xi^3/\pi^2$, which describes the waveguide frequencies of the first mode versus the shift of the oscillation phase in the neighboring fundamental domains of the group of translations.

3. ANOMALOUS PROPERTY. FINE STRUCTURE OF THE SPECTRUM

Theorem 3.1. *The nontrivial anomalous frequencies and the anomalous functions of Problem* $B(0)$ *exist for any lengths of the profiles of a simple knife-type grating; note that the anomalous frequencies belong to the range* $(\pi, 2\pi)$.

Proof. The proof is similar to that of Theorem 2.1.

Estimate from Below. By virtue of Lemma 1.2, the antisymmetry condition for anomalous oscillations with respect to the $y = 1/2$ axis is true, whence follows the estimate from below. It suffices to consider the case of $R = L/2$, for which the inequality $\pi \leqslant \lambda_{\text{NR}}$ holds true.

Estimate from Above. In (2.6), it suffices to choose the components u_{cont} and u_{discont} of the approximate anomalous function satisfying the conditions of Lemma 1.2 in the form

$$u_{\text{cont}} = \sin(2\pi y)\cos(\pi x/2R), \quad u_{\text{discont}} = \begin{cases} æ(y - 1/2)\cos(\pi x/L), & \{|x| < L/2,\ 0 < y < 1\} \\ 0, & \{|x| > L/2,\ 0 < y < 1\} \end{cases},$$

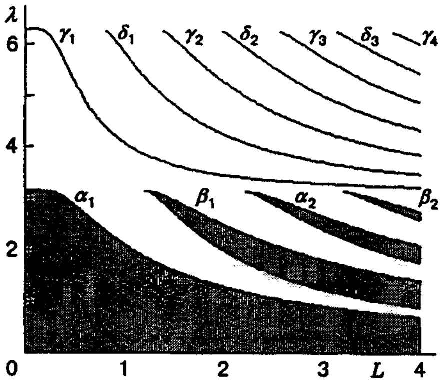

Fig. 4

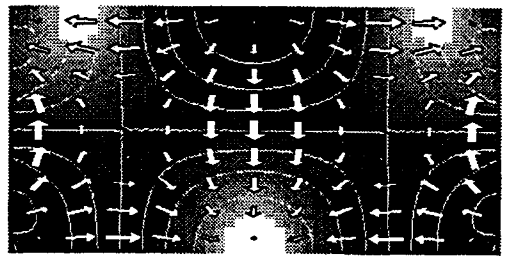

Fig. 5

For large R and small $æ$, the strict inequality $\mu^2(æ, R) < 4\pi^2$ [in the notation of (2.8)] is true. Theorem 3.1 is proved.

Together with the passbands, which belong to the range $(0, \pi^2)$, the spectrum of the operator of Problem B contains a finite set of generalized eigenvalues corresponding to the frequencies of anomalous oscillations.

Theorem 3.2. In the range $(\pi, 2\pi)$, the number $K(L)$ of frequencies of the anomalous modes of oscillations satisfy the relations $\max(1, [L\sqrt{3}]) \leqslant K(L) \leqslant [L\sqrt{3}] + 1$. Here [ ] is the integer part of the corresponding number. If $L\sqrt{3}$ is an integer, we have $(L\sqrt{3} - 1) \leqslant K(L) \leqslant L\sqrt{3}$.

Proof. Let $R = L/2$ and $\lambda_{\mathrm{NR}}^{(k)}$ and $\lambda_{\mathrm{DR}}^{(k)}$ be the eigenfrequencies of Problem $B(0)$ for the domain Ω_1 with the Neumann or Dirichlet conditions at the boundary $|x| = L/2$, respectively. For a certain natural number K, the expressions for the eigenvalues, numbered in ascending order, are as follows:

$$\{\lambda_{\mathrm{NR}}^{(k)}\}_{k=1,\ldots,K+2} = \Big\{\pi, \pi\sqrt{1 + 1/L}, \ldots, \pi\sqrt{1 + (K-1)^2/L}, \pi\sqrt{1 + K^2/L} \geqslant 2\pi\Big\},$$

$$\{\lambda_{\mathrm{DR}}^{(k)}\}_{k=1,\ldots,K} = \Big\{\pi\sqrt{1 + 1/L}, \ldots, \pi\sqrt{1 + (K-1)^2/L}, \pi\sqrt{1 + K^2/L} \geqslant 2\pi\Big\}.$$

Since the number of anomalous frequencies in the range $(\pi, 2\pi)$ cannot be greater than K^* such that $\lambda_{\mathrm{NR}}^{(K^*)} < 2\pi \leqslant \lambda_{\mathrm{NR}}^{(K^*+1)}$ and is not less than K_* such that $\lambda_{\mathrm{DR}}^{(K_*)} < 2\pi \leqslant \lambda_{\mathrm{DR}}^{(K_*+1)}$, the fact that was to be shown follows.

In the corresponding domains, for the γ and δ modes of oscillations the solutions of Problem $B(0)$ have the form

$$u_1(x, y) = \sum_{m=1}^{+\infty} a_m \cos\left[(2m-1)\pi y\right] \left\{ \begin{array}{c} \cosh\left[x\sqrt{(2m-1)^2\pi^2 - \lambda^2}\right] \\ \sinh\left[x\sqrt{(2m-1)^2\pi^2 - \lambda^2}\right] \end{array} \right\},$$

$$u_2(x, y) = \sum_{n=1}^{+\infty} b_n \sin(2\pi n y) \exp\left[-x\sqrt{(2\pi n)^2 - \lambda^2}\,\right], \qquad \left\{ \begin{array}{c} u_3(-x, y) = u_2(x, y) \\ u_3(-x, y) = -u_2(x, y) \end{array} \right\}.$$

After the solutions in this form are substituted into (2.3), one can obtain relations which were discretized and studied numerically by the method of taking into account directly the finite character of the energy [2].

Figure 4 shows the frequency of anomalous oscillations versus the length of the element of a simple grating for the γ and δ modes. For the case of $L = 2$, the field of velocities, the level lines, and the pressure field for the γ_2 mode in the interprofile channel $0 < y < 1$ are shown in Fig. 5. The form of the modes of anomalous oscillations in other interprofile channels is determined by the periodicity condition. The mechanical analog of these modes is the oscillations of several coupled chains of coupled oscillators: the γ_1 mode is synphase oscillations of one chain, the δ_1 mode is synphase oscillations of two coupled chains (the chains are in antiphase), and the γ_2 mode is the synphase oscillations of three coupled chains (two chains oscillate in phase, and one chain in antiphase relative to each other). It is worth noting the great difference between the

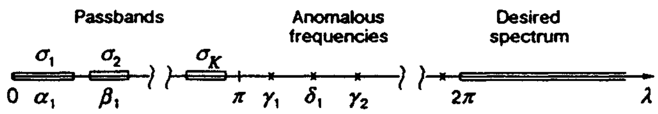

Fig. 6

anomalous oscillations near a grating and the synphase oscillations of the chain of coupled oscillators.

Remark 3.1. The frequency of anomalous oscillations near a simple knife-type grating is greater than the frequency of waveguide oscillations. The frequency of synphase oscillations of the chain of coupled oscillators is smaller than the frequency of waveguide modes.

The results allow one to refine the fine structure of the spectrum of self-conjugate extensions of the Laplace operator, which corresponds to Problem B for a simple knife-type grating. In terms of the dimensionless frequencies, the fine structure of the spectrum of Problem B is shown in Fig. 6. It is necessary to note that, according to the proved theorems for any length of the grating elements, there is always at least one passband (of type α) and one frequency of anomalous oscillations (of type γ).

The passbands of a simple knife-type grating belong to the range $(0, \pi)$, the dimensionless frequencies of anomalous oscillations are $(\pi, 2\pi)$, and the number of the passbands and anomalous frequencies depend on the lengths of the elements of the simple knife-type grating.

Conclusions. (1) It was proved that simple combined and double knife-type gratings always have the waveguide and anomalous properties. The criteria for the existence of these properties for various waveguide modes were obtained. It was shown that there are a finite number of passbands and anomalous modes, and these modes were attributed to the admissible symmetries.

(2) Approximate dispersion relations which describe the propagation of the waves localized in the neighborhood of simple knife-type gratings were obtained and analyzed. The asymptotical form of the dispersion relations was investigated with an infinite increase in the dimensions of the grating elements and a decrease in the shift of the oscillation phase in the adjacent fundamental domains of the group of translations.

(3) The waveguide and anomalous frequencies and the number of waveguide and anomalous modes versus the linear dimensions of the elements of a simple knife-type grating were investigated.

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REFERENCES

1. S. V. Sukhinin, "Effect of a waveguide," *Prikl. Mekh. Tekh. Fiz.*, No. 2, 92–102 (1989).
2. S. V. Sukhinin, "Eigenoscillations near a plate in a channel," *Prikl. Mekh. Tekh. Fiz.*, **39**, No. 2, 78–90 (1998).
3. R. Mittra and S. W. Lee, *Analytical Techniques in the Theory of Guided Waves*, Macmillan (1971).
4. L. Brillouin and M. Parodi, *Propagation des Ondes dans les Milieux Periodiques*, Masson, Paris (1956).
5. M. Reed and B. Simon, *Methods of Modern Mathematical Physics*, Vol. 4, Academic Press (1978).